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Application of Linearization Analysis to Aircraft Dynamics

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A method for determining when a linear model is an acceptable qualitative approximation capable of predicting the dynamic behavior of a nonlinear model is proposed. Relevant results from the qualitative theory of dynamic systems are reviewed. The method, which is intended for use in linear controller design, for reducing the number of nonlinear simulations required to validate controller designs, and in determining the validity of the simulation model in relation to the physical plant, is illustrated by applying it to a F4-J aircraft model. The method successfully determines when linearization is an acceptable qualitative approximation to the dynamic behavior of the nonlinear dynamic model.

Introduction

OWING to the extensive and definitive theory that exists for linear systems, linear models are widely used in engineering analysis and design. Consequently, it is essential to know when linearization is an acceptable approximation. One frequently used method of verifying the linearization approximation is to perform a large number of simulations in the operating region of interest. Consider, however, the danger of relying only on simulation without understanding the underlying system dynamics. Assume a model has 80 variables with each variable having a choice of 10 values. A comprehensive simulation study would require 10^{80} computer simulations, which is impossible.

Carroll and Mehra[1] have proposed a method for analyzing the nonlinear and high-α dynamic behavior of aircraft, which was a bifurcation and catastrophe theory methodology (BACTM). Their paper concentrates on the high-α phenomena such as prestall buffeting, stall, poststall departure, and spin. However, it is not apparent how the results from BACTM are related to the qualitative theory of differential equations of which bifurcation and catastrophe theory are a part.

Using the concept of the set of all equilibrium points in the operating region of interest, called the equilibrium surface,[1] a method is therefore proposed in the following sections for determining whether linearization is an acceptable qualitative approximation to a nonlinear system's dynamic behavior. The proposed method is to determine and analyze the set of all equilibrium points in the operating region of interest. The underlying theory for the analysis of the equilibrium points is based on the qualitative theory of dynamic systems.[2-4] The use of the method and its relation to the underlying theory is illustrated by successfully applying it to a F4-J aircraft model.

Linearization

Since, at present, there exists a definitive theory only for linear dynamic systems, engineers are forced to use linear analysis and design methods for most practical problems. Hence, the question often asked is, "When is linearization an acceptable approximation?" The question as it stands is meaningless within a physical science context since the word approximation needs to be qualified. This problem is overcome by rephrasing the question as, "When is linearization an acceptable *qualitative* approximation?" The adjective qualitative is used to denote the inherent local features and properties of the behavior of a dynamic system about an operating point. A local property is valid in a neighborhood of a point, the size of which is not specified.

The question can best be answered by considering results from the qualitative theory of dynamic systems.[2-4] We consider the local study of dynamic systems, which leads to the study of differential equations in an open subset of Euclidean space $\boldsymbol{R}^n$. In general, the solution of n first-order autonomous nonlinear differential equations

$$\dot{x}(t) = f[x(t)] \tag{1}$$

where $x \in \boldsymbol{R}^n$ is the state vector and $f:\boldsymbol{R}^n \rightarrow \boldsymbol{R}^n$ is the vector field and cannot be found analytically. Hence, the problem of investigating the properties of the solutions of a differential equation from the equation itself arises, and this suggests using available results from the qualitative theory of dynamic systems. The analytical solution or numerical integration of differential equations is not required. One of the key aspects determining the qualitative dynamic behavior of nonlinear dynamic systems is the nature of the plant steady-state behavior[5] as $t \rightarrow \infty$. This leads to an investigation of the nonlinear system dynamic behavior in the neighborhood of the equilibrium points[5] defined by

$$f[x(t)] = 0 \tag{2}$$

The linearization of the nonlinear vector field f at the equilibrium point $x = x^*$ is given by

$$\delta\dot{x}(t) = F\delta x(t) \tag{3}$$

where $\delta x(t) = x(t) - x^*$ and F is the Jacobian matrix of f at x^*.

The important question that arises now is, "What can be inferred about the solutions of the nonlinear dynamic system [Eq. (1)] from the solutions of the linearized dynamic system [Eq. (3)] in the neighborhood of the equilibrium points?" Since we are only concerned with a qualitative understanding of the local behavior of a dynamic system in the neighborhood of an equilibrium point, we only need topological information about the local nonlinear vector field.[2] This leads to a key result,[2] denoted as Theorem 1, namely, that if F has no eigenvalues with zero real part, then the local dynamic behavior of the nonlinear and linear dynamic systems in the neighborhood of the equilibrium point x^* are similar. In other words, in the neighborhood of the equilibrium point x^*, if the nonlinear system exhibits exponentially stable dynamic behavior then so will the linearized dynamic system exhibit this same type of behavior. Their respective behaviors are said to be qualitatively similar.

When using linear models to approximate nonlinear systems, Theorem 1 is implicitly assumed in the linearization

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process. When the linearized model has eigenvalues with real part equal to zero, there is a potential problem in assuming that the linear model qualitatively represents the nonlinear model dynamic behavior in the neighborhood of the equilibrium point, and the bifurcation theory must be used to analyze this case.

Definition 1: If the Jacobian matrix F has no eigenvalues with zero real part, then the equilibrium point x^* is called a hyperbolic equilibrium point. Hyperbolic and nonhyperbolic equilibrium points are called nondegenerate and degenerate equilibrium points, respectively.

Equation (1) represents a system with a vector field f with fixed parameter values, called a specific vector field. However, most physical plants have vector fields that change due to variation in one or more parameters of the vector field. The collection of specific vector fields for all values of the parameters is called a family of vector fields. If the vector field variation is due to one parameter, then one refers to a one-parameter family of vector fields. The important point is that degenerate equilibrium points are not removable by an arbitrarily small perturbation of the vector field parameters when a whole family of vector fields is considered.[4] We will only consider a one-parameter family of vector fields, which is the only case comprehensively studied in the literature. A one-parameter family of vector fields or, in local terms, a one-parameter family of differential equations is defined by

$$\dot{x}(\alpha,t) = f[x(\alpha,t),\alpha] \tag{4}$$

where $x \in R^n$, $f{:}R^n \times R \to R^n$ and $\alpha \in R$ is a parameter, usually a control input in control applications, whose value is varied. All other vector field parameters are constant. The equilibrium points of Eq. (4) must satisfy

$$f[x(\alpha,t),\alpha] = 0 \tag{5}$$

Equation (5) defines the set

$$S = \{x \in R^n, \alpha \in R \,|\, f[x(\alpha,t),\alpha] = 0\} \tag{6}$$

of equilibrium points of the one-parameter family of differential equations. The linearization of the nonlinear vector field f at the equilibrium point $x = x^*$, $\alpha = \alpha^*$ is given by

$$\delta\dot{x}(\alpha,t) = F\delta x(\alpha,t) \tag{7}$$

where $\delta x(\alpha,t) = x(\alpha,t) - x^*(\alpha^*)$ and F is the Jacobian matrix of f at x^*, $\alpha^* \in S$. The value of the parameter α for which the equilibrium point (x,α) of S is degenerate is called the bifurcation value and the point (x,α) is called a bifurcation point.

The question now asked is, "What happens to the equilibrium points given by Eq. (5) as the parameter α is varied?" The answer is given by the following two bifurcation theorems.[4] The first result, denoted as Theorem 2, states that if the eigenvalues of the Jacobian matrix F in Eq. (7) do not have zero real part, then the vector field f is continuous with respect to α. The second result, denoted as Theorem 3, states that if for a nonlinear dynamic system given by Eq. (4), α is a bifurcation value corresponding to a degenerate equilibrium point of S, then the Jacobian matrix F of the linearized system either has a real zero eigenvalue or a conjugate pair of complex eigenvalues with zero real part. In addition, these are the only degenerate equilibrium points that cannot be removed by an arbitrarily small perturbation of the vector field parameters.

In basic terms, Theorem 3 means that, for a hyperbolic or nondegenerate equilibrium point of the nonlinear system, the nonlinear dynamic system and its linearized approximation [Eq. (7)] have qualitatively similar behavior in the neighborhood of the hyperbolic equilibrium point. At a degenerate equilibrium point, Theorem 3 states that there are two possible cases. First, as the parameter α is varied and a real eigenvalue of the linearized dynamic system crosses the imaginary axis, a real bifurcation occurs. As α varies through the real bifurcation point, the number of equilibrium solutions of Eq. (5) changes for a given set of model parameter values. The second case occurs when a pair of complex conjugate eigenvalues of the linearized system cross the imaginary axis as the parameter α is varied. This results in the occurrence of a Hopf bifurcation, named after E. Hopf, who in 1942 first proved his well-known theorem. The Hopf Bifurcation Theorem[6] states that, as the parameter α is varied and a pair of complex conjugate eigenvalues of the linearized system crosses the imaginary axis, a Hopf bifurcation occurs in the neighborhood of the equilibrium point. A Hopf bifurcation exhibits limit cycle dynamic behavior. To summarize, in the neighborhood of a bifurcation point, the nonlinear and linear dynamic systems do not have qualitatively similar dynamic behavior.

The system global dynamic behavior can be determined by analyzing the system qualitative dynamic behavior at each equilibrium point over the entire equilibrium surface. The equilibrium surface comprises the set S of all equilibrium points of the dynamic system in the operating region of interest.

When the parameter α is considered as a system input $u(t)$, the equilibrium surface of the more common engineering model formulation

$$\dot{x}(t) = f[x(t),u(t)] \tag{8}$$

that represents the physical plant in the input region of interest is determined. Here, $x(t) \in R^n$ are the states, and the model input $u(t) \in R$ is the bifurcation parameter analogous to the parameter α in Eq. (4). For the case where $u(t) \in R^m$, to ensure that we have a one-parameter family of differential equations, one of the model inputs is selected as the bifurcation parameter and the other model inputs are considered as model parameters with fixed values. The question of what happens when, due to model parameter uncertainty, we analyze a model [Eq. (8)] that is not exactly the same as the real physical system is not addressed in this paper, but will be considered in a future paper.

To investigate the region where linearization is a good qualitative approximation to the nonlinear model, an equilibrium surface of the system model represented by Eq. (8) in the operating region of interest $[u(t)_{\min} \le u(t) \le u(t)_{\max}]$ is determined and the equilibrium points are analyzed using Theorems 1, 2, and 3.

Example of Linearization Analysis

The previous ideas on linearization are applied to a F4-J nonlinear aircraft model whose equations of motion in body axes[7] (see the Appendix) are given by

$$\dot{\alpha} = Q - \tan\beta\,(P\cos\alpha + R\sin\alpha) + [mg\cos\Theta\cos\Phi\cos\alpha + mg\sin\Theta\sin\alpha - T\sin(\alpha+\xi) - L]/(mV_T\cos\beta) \tag{9}$$

$$\dot{\beta} = P\sin\alpha - R\cos\alpha + (mg\sin\Theta\cos\alpha\sin\beta + mg\cos\Theta\sin\Phi\cos\beta - mg\cos\Theta\cos\Phi\sin\alpha\sin\beta - T\sin\beta\cos(\alpha+\xi) + D\sin\beta + S\cos\beta]/(mV_T) \tag{10}$$

$$\dot{V}_T = [mg\cos\Theta\sin\Phi\sin\beta - mg\sin\Theta\cos\alpha\cos\beta + mg\cos\Theta\cos\Phi\sin\alpha\cos\beta + T\cos\beta\cos(\alpha+\xi) - D\cos\beta + S\sin\beta]/m \tag{11}$$

$$\dot{P} = [QR\{(I_Y - I_Z)I_Z - I_{XZ}^2\} + PQ\{(I_X - I_Y + I_Z)I_{XZ}\} + NI_{XZ} + L_rI_Z]/(I_XI_Z - I_{XZ}^2) \tag{12}$$

$$\dot{Q} = [PR(I_Z - I_X) + (R^2 - P^2)I_{XZ} + M]/I_Y \tag{13}$$

$$\dot{R} = [QR\{(I_Y - I_Z - I_X)I_{XZ}\} + PQ\{(I_X - I_Y)I_X + I_{XZ}^2\} + L_r I_{XZ} + NI_X]/(I_X I_Z - I_{XZ}^2) \tag{14}$$

where $m = 16{,}820$ kg, $g = 9.81$ m/s^2, $I_X = 32{,}340$ kg/m^2, $I_Y = 173{,}000$ kg/m^2, $I_Z = 198{,}000$ kg/m^2, $I_{XZ} = 3000$ kg/m^2, $T = 50{,}000$ N, and $\xi = 0$ rad. The lift, side, and drag aerodynamic forces, L, S, and D, respectively, are calculated in terms of

$$L = 0.5\rho V_T^2 s C_L$$

$$S = 0.5\rho V_T^2 s C_Y$$

$$D = 0.5\rho V_T^2 s C_D \tag{15}$$

where ρ is the air density; $s = 49$ m^2 is the wing area; C_L, C_Y, and C_D are dimensionless coefficients that are tabular functions of α, δ_{stab}, δ_a, and δ_r. The δ_{stab}, δ_a, and δ_r are the elevator, aileron, and rudder controls, respectively. The rolling, pitching, and yawing aerodynamic moments L_r, M, and N, respectively, are calculated in terms of

$$L_r = 0.5\rho V_T^2 sbC_l$$

$$M = 0.5\rho V_T^2 s\bar{c}C_m$$

$$N = 0.5\rho V_T^2 sbC_n \tag{16}$$

where $b = 11.8$ m is the wing span; $\bar{c} = 4.9$ m is the mean wing chord; C_l, C_m, and C_n are dimensionless coefficients that are tabular functions of α, δ_{stab}, δ_a, and δ_r. The aerodynamic coefficient tabular data for the F4-J aircraft were taken from Mitchell et al.[7]

Equilibrium Surface of Aircraft Model

The nonlinear aircraft equations of motion are in the form of Eq. (4) with one of the control inputs taken as the bifurcation parameter. The Jacobian matrix of Eqs. (9–14) yields the linearized F4-J aircraft model in the form of Eq. (7) given by

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{v}_T \\ \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} g_{11} & \cdots & g_{18} \\ \vdots & & \vdots \\ g_{81} & \cdots & g_{88} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ v_T \\ p \\ q \\ r \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} gc_{11} & \cdots & gc_{13} \\ \vdots & & \vdots \\ gc_{81} & \cdots & gc_{83} \end{bmatrix} \begin{bmatrix} \delta_{\text{stab}} \\ \delta_a \\ \delta_r \end{bmatrix} \tag{17}$$

where α, β, v_T, p, q, r, ϕ, and θ are the perturbed states from a nominal operating point; δ_{stab}, δ_a, and δ_r are the elevator, aileron, and rudder controls, respectively.

We now show that the aircraft model is represented as a one-parameter family of differential equations. We will investigate the longitudinal dynamic behavior of the aircraft. Hence, δ_{stab} is taken as the input $u(t)$ analogous to Eq. (8).

Table 1 Eigenvalues of the linearized aircraft model at three equilibrium points corresponding to three values of δ_{stab}

$\delta_{\text{stab}} = -7.8$ deg	$\delta_{\text{stab}} = -8.5$ deg	$\delta_{\text{stab}} = -10.4$ deg
$-0.37 + j1.11$	$-0.36 + j1.12$	$-0.37 + j1.14$
$-0.37 - j1.11$	$-0.36 - j1.12$	$-0.37 - j1.14$
$-0.037 + j0.13$	$-0.042 + j1.13$	$-0.051 + j1.14$
$-0.037 - j0.13$	$-0.042 - j1.13$	$-0.051 - j1.14$
$\mathbf{-0.053 + j1.25}$	$\mathbf{0.0 + j1.1}$	$\mathbf{0.23 + j0.61}$
$\mathbf{-0.053 - j1.25}$	$\mathbf{0.0 - j1.1}$	$\mathbf{0.23 - j0.61}$
$-0.26 + j0.0$	$-0.44 + j0.0$	$-1.0 + j0.0$
$-0.35 + j0.0$	$-0.3 + j0.0$	$-0.52 + j0.0$

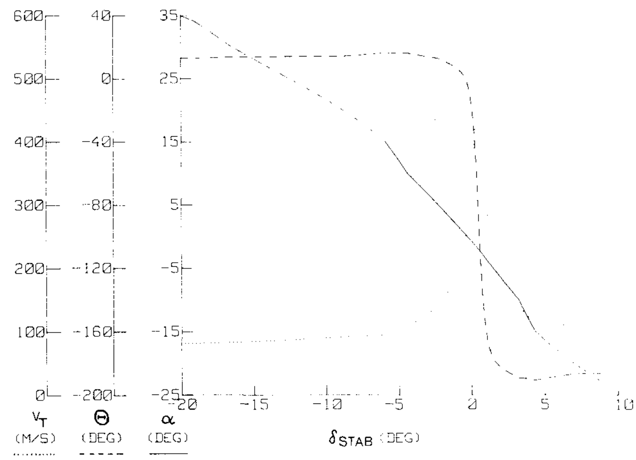

Fig. 1 Equilibrium surface diagram: $h = 4572$ m; $\delta_a = 0$ deg; $\delta_r = 0$ deg.

The other controls δ_a and δ_r that control the aircraft lateral dynamic behavior are set to zero and considered fixed parameter values, as discussed in the section on linearization. The equilibrium surface is determined by solving Eq. (5) for the aircraft equations of motion. The equilibrium diagram for the aircraft model given by Eqs. (9–16) for an altitude of 4572 m (15,000 ft), $\delta_a = 0$, $\delta_r = 0$, and $-20 \le \delta_{\text{stab}} \le 10$ deg is shown in Fig. 1. In Fig. 1, the aircraft velocity and pitch angle are sensitive to δ_{stab} in the region $-2 < \delta_{\text{stab}} < 4$ deg due to the aerodynamic coefficient data used.[7]

Analysis of Equilibrium Points

Having determined the equilibrium surface (Fig. 1), the theory presented in the section on linearization is applied to analyze the equilibrium points. An analysis of the eigenvalues of the linearized models [Eq. (7)] at all of the equilibrium points of the equilibrium surface clearly indicates the region where a pair of complex conjugate eigenvalues of the linearized aircraft model [Eq. (17)] cross the imaginary axis as the elevator δ_{stab} is changed from -7.8 to -10.4 deg, as given in Table 1. As δ_{stab} is changed from -7.8 to -10.4 deg, from Table 1 it is apparent that there is a value of δ_{stab} at which the linearized model will have purely imaginary eigenvalues. This equilibrium point occurs at $\delta_{\text{stab}} = -8.5$ deg. From Theorem 2 and Definition 1, all of the equilibrium points defined by Eq. (6) for $-20 \le \delta_{\text{stab}} \le 10$ deg are nondegenerate except for the degenerate equilibrium point at $\delta_{\text{stab}} = -8.5$ deg (Table 1). Using Theorem 3, the degenerate equilibrium (bifurcation) point at $\delta_{\text{stab}} = -8.5$ deg cannot be removed by an arbitrarily small perturbation of the model parameter values. As discussed in the section on linearization, at this type of degenerate equilibrium point, a limit cycle occurs in the aircraft dynamic behavior.

By Theorem 1, in the neighborhood of the equilibrium point at $\delta_{\text{stab}} = -8.5$ deg, linearization may not be a good qualitative approximation to the aircraft nonlinear dynamic behavior. Hence, simulation is required to determine both the dynamic behavior and the extent of the neighborhood for which this dynamic bahavior occurs. Likewise, by Theorem 1, in the regions for $\delta_{\text{stab}} > -8.5$ deg and for $\delta_{\text{stab}} < -8.5$ deg, both outside the neighborhood of the degenerate equilibrium point, linearization is a good qualitative approximation to the aircraft nonlinear dynamic behavior. Hence, simulation is not required to determine the dynamic behavior of the nonlinear system in this region.

Simulation Results

The previous conclusions were now verified by simulation using several case studies. Consider an elevator δ_{stab} input with $\delta_a = \delta_r = p = r = \phi = 0$, wherein only the aircraft longitudinal

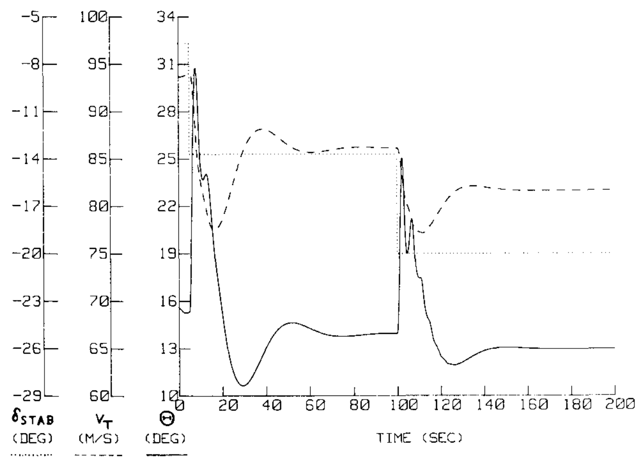

Fig. 2 Nonlinear simulation for a δ_{stab} change from -6.7 to -13.7 deg and thereafter to -20 deg: $h = 4572$ m; $\delta_a = \delta_r = \beta = p = r = \phi = 0$.

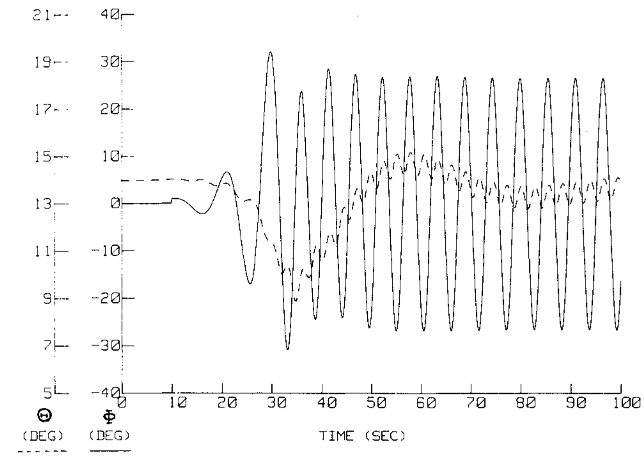

Fig. 4 Nonlinear simulation for $\delta_{stab} = -10.4$ deg with a perturbation in ϕ of 1.0 deg after 10 s: $h = 4572$ m; $\delta_a = \delta_r = \beta = p = r = 0$.

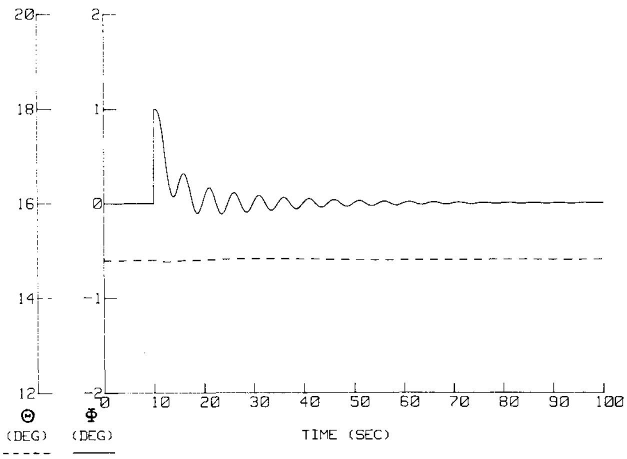

Fig. 3 Nonlinear simulation for $\delta_{stab} = -7.8$ deg with a perturbation in ϕ of 1.0 deg after 10.0 s: $h = 4572$ m; $\delta_a = \delta_r = \beta = p = r = 0$.

dynamics [Eqs. (9), (11), (13)] are excited. The lateral dynamics [Eqs. (10), (12), (14)] are decoupled from the longitudinal dynamics, which is a common assumption made for most aircraft control design using linearized models. The linearized model eigenvalues for $\delta_{stab} = -7.8$ and -10.4 deg given in Table 1 predict exponentially decaying oscillatory longitudinal dynamic behavior, which was verified by simulation, as shown in Fig. 2. Thus, as stated by Theorem 1, the aircraft nonlinear and linear longitudinal dynamic behavior is qualitatively similar at the equilibrium points corresponding to $\delta_{stab} = -7.8$ and -10.4 deg, and linearization is a good qualitative approximation in this situation.

The linearized model eigenvalues for $\delta_{stab} = -7.8$ deg given in Table 1 predict exponentially decaying oscillatory lateral dynamic behavior, which was verified by simulation with the aircraft roll attitude ϕ perturbed due to a gust or turbulence, as shown in Fig. 3. The linearized model eigenvalues for $\delta_{stab} = -10.4$ deg given in Table 1 predict exponentially increasing oscillatory lateral dynamic behavior. However, simulation with the aircraft roll attitude ϕ perturbed due to a gust or turbulence shows, in Fig. 4, that for $\delta_{stab} = -10.4$ deg the aircraft exhibits a stable limit cycle. Since a degenerate equilibrium (bifurcation) point exists at $\delta_{stab} = -8.5$ deg for the lateral dynamic case, Theorem 1 states that the aircraft nonlinear and linear dynamic behavior is not qualitatively similar in the neighborhood of this equilibrium point and linearization is not a good qualitative approximation in this situation. Simulation verified the conclusions of Theorem 1 at the degenerate equilibrium point ($\delta_{stab} = -8.5$ deg) and confirmed that the equilibrium points at $\delta_{stab} = -7.8$ and -10.4 deg are outside and within the neighborhood of the equilibrium point $\delta_{stab} = -8.5$ deg, respectively, wherein the dynamic behavior is similar.

In addition, simulation confirmed the results of Theorem 3 that there exists a degenerate equilibrium point ($\delta_{stab} = -8.5$ deg) that cannot be removed by an arbitrarily small perturbation of the vector field parameters and that this degenerate equilibrium point must correspond to the eigenvalues of the Jacobian of the linearized model being on the imaginary axis.

Hence, the analysis and interpretation of the model equilibrium points using Theorems 1, 2, and 3 gives an indication of the neighborhood in which linearization is a good qualitative approximation to the model nonlinear dynamic behavior. The results can be verified by simulation. In addition, the analysis of the equilibrium surface also indicates the regions, notably at degenerate equilibrium (bifurcation) points, where simulation should be done to validate the model against the physical plant and to verify controller designs. This greatly reduces the number of simulations required in engineering analysis and design.

An interesting outcome of the F4-J aircraft application indicated the danger of assuming decoupled aircraft longitudinal and lateral dynamics for control design in the operating region $\delta_{stab} \leq -8.5$ deg. A slight perturbation in any lateral parameter such as ϕ caused a limit cycle in the aircraft longitudinal dynamics (Fig. 4) that did not occur in the decoupled case (Fig. 2). This would not have been seen easily from using simulation blindly and is one of the advantages of using this approach to the analysis of nonlinear systems.

Finally, it should be mentioned that, if the effects of the lateral model inputs δ_a and δ_r, are required, then, in addition to the equilibrium surface diagram shown in Fig. 1, the equilibrium surface diagrams with each of δ_a and δ_r, as the bifurcation parameter can be determined. In this way, the complete equilibrium surface for all values of δ_{stab}, δ_a, and δ_r will be determined. In addition, the equilibrium diagrams at different altitudes to account for air density variation with altitude need to be determined.

Conclusions

Results from the qualitative theory of dynamic systems can be used to effectively determine when linearization is a good qualitative approximation to the behavior of a nonlinear dynamic model. These results were successfully applied to a F4-J aircraft model and verified using simulation. Analysis of the equilibrium points can also be used to determine the region of operation where simulation is necessary to verify the model's dynamic behavior. In particular, the number of simulations required to validate models against the physical plant and to verify controller designs is greatly reduced.

Appendix: Aircraft Equations of Motion

From the standard aircraft equations of motion in terms of body axes[8]

$$m(\dot{U} + QW - RV + g \sin \Theta) = T \cos \xi - D \cos \alpha + L \sin \alpha \tag{A1}$$

$$m(\dot{V} + RU - PW - g \cos \Theta \sin \Phi) = S \tag{A2}$$

$$m(\dot{W} + PV - QU - g \cos \Theta \cos \Phi)$$

$$= -T \sin \xi - D \sin \alpha - L \cos \alpha \tag{A3}$$

$$\dot{P} I_X - \dot{R} I_{XZ} + QR(I_Z - I_Y) - PQI_{XZ} = L_r \tag{A4}$$

$$\dot{Q} I_Y + PR(I_X - I_Z) + (P^2 - R^2) I_{XZ} = M \tag{A5}$$

$$\dot{R} I_Z - \dot{P} I_{XZ} + PQ(I_Y - I_X) - QRI_{XZ} = N \tag{A6}$$

where m is the aircraft mass; g the acceleration due to gravity; I_X, I_Y, I_Z, and I_{XZ} the moments and product of inertia; T the thrust; ξ the angle between the engine thrust line and the aircraft X body axis; Θ and Φ the Euler angles of pitch and roll, respectively; U, V, and W the aircraft velocities along the X, Y, and Z body axes, respectively; α and β the angle of attack and sideslip, respectively; P, Q, and R the aircraft roll, pitch, and yaw rates, respectively; and L, S, D, L_r, M, and N the aerodynamic lift, side, drag, rolling, pitching, and yawing forces and moments, respectively.

Defining α, β, and the resultant aircraft velocity V_T in terms of U, V, and W, one obtains

$$\alpha = \tan^{-1} W/U \tag{A7}$$

$$\beta = \sin^{-1} V/V_T \tag{A8}$$

$$V_T = \sqrt{U^2 + V^2 + W^2} \tag{A9}$$

where

$$U = V_T \cos \alpha \cos \beta \tag{A10}$$

$$V = V_T \sin \beta \tag{A11}$$

$$W = V_T \sin \alpha \cos \beta \tag{A12}$$

Differentiating Eqs. (A7–A9), one obtains

$$\dot{\alpha} = \left(\frac{\dot{W}U - W\dot{U}}{U^2} \right) \Big/ \sec^2 \alpha \tag{A13}$$

$$\dot{\beta} = \left(\frac{\dot{V}V_T - V\dot{V}_T}{V_T^2} \right) \Big/ \cos \beta \tag{A14}$$

$$\dot{V}_T = \left(\frac{U\dot{U} + V\dot{V} + W\dot{W}}{V_T^2} \right) \tag{A15}$$

Substituting Eqs. (A1–A3) and (A10–A12) into Eqs. (A13–A15) for U, V, W, $\dot{U}$, $\dot{V}$, $\dot{W}$ and solving explicitly for $\dot{\alpha}$, $\dot{\beta}$, $\dot{V}_T$, one finally obtains Eqs. (9–11).

Solving Eqs. (A4–A6) explicitly for $\dot{P}$, $\dot{Q}$, and $\dot{R}$, one obtains Eqs. (12–14).

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